SOME EXTREMAL PROBLEMS FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY
ROMAN BOUTELLIER AND ALBERT PFLUGER

ABSTRACT

A variational method is developed within the class of functions of boundary rotation not exceeding $k\pi$ which is based on the fact that the set of representing measures μ is convex. It shows that an extremal problem related to a functional with Gâteaux derivative and some constraints leads to extremal measures μ_0 with finite support. The positive and negative part of a μ_0 is located at points where a function J (depending on μ_0) reaches its maximum and minimum respectively. The method is tested successfully on various problems.

1. Introduction

For $k \geqq 2$ let V_k denote the class of locally univalent functions,

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$

in the unit disc $\mathbf{D} = \{|z| < 1\}$ which have boundary rotation at most $k \cdot \pi$, i.e.

$$\lim_{r \to 1} \int_0^{2\pi} d_\theta \left| \arg \frac{d}{d\theta} f(re^{i\theta}) \right| \leqq k \cdot \pi.$$

A function f with $f(0) = 0$, $f'(0) = 1$, belongs to V_k if and only if it is a solution of the differential equation

$$(1) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2} \int \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} d\mu(\theta),$$

where μ is a real measure with support on the interval $[0, 2\pi]$ (or sometimes on another interval of length 2π, or equivalently on the unit circle $\{|z| = 1\}$), such that

$$(2) \qquad \int d\mu = 2, \qquad \int |d\mu| \leqq k.$$

Received August 5, 1980 and in revised form November 24, 1980

This result is due to Paatero [8]. In the sequel we shall denote the set of all such measures μ by M_k. Integrating the above differential equation we get the integral representation

$$f'(z) = \exp \int -\log(1 - e^{i\theta}z)d\mu. \tag{3}$$

If the positive part of μ is concentrated on $\theta = 0$ and its negative part on $\theta = \pi$, we get the function F_k,

$$F_k'(z) = \frac{(1+z)^{k/2-1}}{(1-z)^{k/2+1}},$$

which is extremal for many problems in V_k: For example F_k maximizes Re a_n for all n within V_k as was shown by D. A. Brannan, J. G. Clunie and W. E. Kirwan, and by D. Aharonow and S. Friedland in 1973 (Ann. Acad. Sci. Fenn. Sci. AI, 523 and 524).

In the paper we introduce side restrictions of the form

$$a_2 = \cdots = a_m = 0.$$

As we shall see, this gives rise to certain symmetries. The problems, restricted and unrestricted, are solved by a variational method which is an elaboration of the simple fact that M_k is convex. The method may be applied to other kinds of extremal problems involving such measures.

2. Standard variations (μ_ε-variations)

We denote by H the space of holomorphic functions on the unit disc, provided with the topology of locally uniform convergence. Let ϕ be a complex-valued functional on V_k which is Gâteaux differentiable on V_k, i.e. for each $f \in V_k$ there exists a continuous linear functional $L = L_f$ on H, such that

$$\phi(f + \varepsilon g) = \phi(f) + \varepsilon \cdot L(g) + o(\varepsilon), \qquad \varepsilon \to 0, \tag{4}$$

for each $g \in H$ with $f + \varepsilon g \in V_k$, where $o(\varepsilon)/\varepsilon$ converges to zero uniformly in g on a compact subset of H. L_f is called the Gâteaux derivative of ϕ at f. Since $g(0) = 0$, there exists on H a continuous linear functional L' such that $L(g) = L'(g')$ for functions g appearing in (4). Thus we may write (4) in the form

$$\phi(f + \varepsilon g) = \phi(f) + \varepsilon \cdot L'(g') + o(\varepsilon). \tag{5}$$

For each μ_0 of M_k we define a class of standard variations by

$$\mu_\varepsilon = (1 - \varepsilon) \cdot \mu_0 + \varepsilon \cdot \mu = \mu_0 + \varepsilon(\mu - \mu_0),$$

where μ is in M_k and $0 \leqq \varepsilon \leqq 1$. Since M_k is convex, this is a variation within M_k.

Let by the integral (3) f'_ε correspond to μ_ε; then f'_ε is the standard variation of f'_0 corresponding to the arbitrarily chosen μ of M_k:

$$f'_\varepsilon(z) = \exp \int -\log(1 - e^{i\theta} z) d(\mu_0 + \varepsilon(\mu - \mu_0))$$

$$= f'_0(z) - \varepsilon \cdot f'_0(z) \int \log(1 - e^{i\theta} z) d(\mu - \mu_0) + O(\varepsilon^2).$$

By the integration we get the standard variation of f_0:

$$f_\varepsilon = f_0 + \varepsilon(g + O(\varepsilon)) + o(\varepsilon),$$

where

$$g'(z) = -\int f'_0(z) \log(1 - e^{i\theta} z) d(\mu - \mu_0). \tag{6}$$

Thus, from (5) it follows

$$\begin{aligned} \phi(f_\varepsilon) &= \phi(f_0) + \varepsilon L(g + O(\varepsilon)) + o(\varepsilon) \\ &= \phi(f_0) + \varepsilon L'(g') + o(\varepsilon). \end{aligned} \tag{7}$$

L' has the well known representation (see e.g. [13])

$$L'(g') = \int_C g'(\xi) d\psi(\xi), \tag{8}$$

where ψ is a finite Borel measure on a compact subset C of $\mathbf{D}$. Thus, if f_0 maximizes $\operatorname{Re} \phi$ in V_k it follows from (7) that $\operatorname{Re} L'(g') \leqq 0$ for all g' of (6); hence by (8)

$$\operatorname{Re} \left\{ \int_C -f'_0(\xi) \log(1 - e^{i\theta} \xi) d\psi(\xi) d(\mu - \mu_0) \right\} \leqq 0.$$

Define now

$$J(\theta) = -\operatorname{Re} \int_C f'_0(\xi) \log(1 - e^{i\theta} \xi) d\psi(\xi) \tag{9}$$

to get

$$\int J(\theta) d(\mu - \mu_0) \leqq 0 \tag{10}$$

for all $\mu \in M_k$.

This is a necessary condition for f_0 to maximize $\operatorname{Re}\phi$ in V_k. J is called the *indicator* corresponding to f_0 and L.

In order to get more information from condition (10), let us recall that any two probability measures ν and ν' on $[0,2\pi]$ define by

$$\mu = (k/2+1)\cdot\nu - (k/2-1)\cdot\nu' \tag{11}$$

a measure of M_k and that conversely each μ of M_k is representable in this form. ($(k/2+1)\nu$ and $(k/2-1)\nu'$ need not be the upper and lower variation of μ, see [10].)

Let now ν_0 and ν_0' be two probability measures representing the extremal measure μ_0 in (11). It follows from inequality (10) by choosing first ν arbitrarily and $\nu' = \nu_0'$, second $\nu = \nu_0$ and ν' arbitrarily, that

$$\int J(\theta)d(\nu-\nu_0) \leqq 0 \quad \text{and} \quad \int J(\theta)d(\nu'-\nu_0') \geqq 0$$

for all probability measures ν and ν'. These two conditions imply that ν_0 is supported by those points of $[0,2\pi]$, where J reaches its maximum, and ν_0' by those points where J attains its minimum.

Therefore, it is of interest to know in which cases $J(\theta)$ is not a constant, because then, $J(\theta)$ being real-analytic, the sets $\operatorname{supp}\nu_0$ and $\operatorname{supp}\nu_0$ have to be finite.

The claim is that $J(\theta)$ is constant (hence zero) if and only if the Gâteaux derivative L is constant on V_k. This may be seen as follows. J being constant implies by (9) $L'(z^n f_0'(z)) = 0$, $n = 1,2,\cdots$. Hence $L'(gf_0') = 0$ for each g of H with $g(0) = 0$, and because of $f_0'(z) \neq 0$ in $\mathbf{D}$ it follows $L'(f - f(0)) = 0$ for all f of H. From $L(f - f(0)) = L'(f')$ we conclude that there are constants a and b such that

$$L(f) = a\cdot f(0) + b\cdot f'(0), \qquad f \in H,$$

and this implies that L is constant on V_k.

The converse is trivial because for each θ the analytic function $f_0'(z)\log(1-ze^{i\theta})$ is the locally uniform limit of functions of V_k', the derivatives of V_k.

Therefore, if L is not constant on V_k, the indicator J has finitely many (absolute) maxima carrying ν_0, and finitely many (absolute) minima carrying ν_0'. Hence, ν_0 and ν_0' have disjoint supports and this implies that $\mu_0^+ = (k/2+1)\nu_0$ and $\mu_0^- = (k/2-1)\nu_0'$ are the upper and lower variation of μ_0. Thus the extremal function f_0 has boundary rotation precisely k.

Summarizing we have

LEMMA 1. *Let ϕ be a functional on V_k admitting at each function of V_k a Gâteaux derivative which is not constant on V_k. Let f_0 maximize* $\operatorname{Re}\phi$ *within V_k and let J be the indicator* (9) *corresponding to f_0 and the Gâteaux derivative L at f_0. Then, if μ_0 is the representing measure of f_0, the inequality*

$$\int J(\theta)d(\mu-\mu_0)\leqq 0$$

is satisfied for each μ of M_k. This extremality condition holds if and only if the positive and the negative part of μ_0 is carried by those disjoint finite subsets of $[0,2\pi]$ on which J reaches its maximum and minimum respectively:

$$\begin{aligned}&\operatorname{supp}\mu_0^+\subset\{\theta:J(\theta)=\max J\},\\ &\operatorname{supp}\mu_0^-\subset\{\theta:J(\theta)=\min J\}.\end{aligned}\tag{12}$$

Moreover, the boundary rotation of f_0 is precisely k.

REMARKS. (1) Support points of V_k are extremals of linear functionals which are not constant on V_k. Lemma 1 immediately implies that each support point of V_k has boundary rotation k and has a representing measure with finite support.

(2) Lemma 1 sharpens a result given by Kirwan and Schober [2] which says that the extremal function f_0 has a representing measure with finite support and total variation $\leqq k$.

(3) Although at first glance condition (12) looks quite promising for solving extremal problems one has to be aware that the indicator J depends on the unknown function f_0 and this dependence can have a severe impact on the number and the location of the maxima and minima of J. Nevertheless it is possible to come to a complete determination of the extremal measure in quite a few cases (cf. [1] for the case of starlike functions).

Now we add constraints to our extremal problem by introducing on M_k the functionals

$$x_j(\mu)=\int X_j(\theta)d\mu,\qquad j=1,\cdots,n,$$

where the $X_j(\theta)$ are real-valued functions on $[0,2\pi]$. For fixing the situation we suppose that they are continuous and that no non-trivial linear combination of them is constant. Therefore the set

$$A=\{(x_1(\mu),\cdots,x_n(\mu))\mid\mu\in M_k\},\tag{13}$$

which is a compact and convex subset of $\mathbf{R}^n$ contains interior points, i.e. A is a convex body in $\mathbf{R}^n$.

Let $J(\theta)$ be a real-valued and continuous function on $[0,2\pi]$ such that $J(\theta)$, $X_1(\theta),\ldots,X_n(\theta)$ and the constant 1 are linearly independent. Choose a point $\xi=(\xi_1,\cdots,\xi_n)$ in the interior of A and consider the problem of maximizing on M_k the functional

$$y(\mu)=\int J(\theta)d\mu$$

under the side conditions $x_j(\mu)=\xi_j$, $j=1,\cdots,n$ or shortly $x(\mu)=\xi$.

LEMMA 2. *Let the function $J, X_1,\cdots,X_n$ satisfy the above conditions and consider, with the above notations, the problem of maximizing $y(\mu)$ over M_k under the constraints $x(\mu)=\xi$, where ξ is an inner point of A. Then there are real numbers $\lambda_1,\cdots,\lambda_n$ (Lagrangian multipliers) such that*

$$\max_{M_k}(y(\mu)+\lambda_1x_1(\mu)+\cdots+\lambda_nx_n(\mu))=\max\{y(\mu)\,|\,\mu\in M_k,\, x(\mu)=\xi\}.$$

PROOF. The set

$$B=\{(y(\mu),x_1(\mu),\cdots,x_n(\mu))\}_{\mu\in M_k}$$

is a convex body in $\mathbf{R}^{n+1}$. Because ξ is in the interior of A, the straight line $x=\xi$ in $\mathbf{R}^{n+1}$ intersects B in a segment not reducing to a point. Let $\bar{y}$ be its upper endpoint, i.e.

$$\bar{y}=\max\{y\,|\,x=\xi,(y,x)\in B\}.$$

Obviously $(\bar{y},\xi)$ is on the boundary of B and there is at least one supporting hyperplane to B through $(\bar{y},\xi)$.

Let $\lambda_0,\lambda_1,\cdots,\lambda_n$ be a normal vector such that

$$\lambda_0(y-\bar{y})+\lambda_1(x_1-\xi_1)+\cdots+\lambda_n(x_n-\xi_n)<0$$

for all interior points (y,x) of B. Let (y_1,ξ) be such an interior point which obviously exists. Then it follows $\lambda_0(y_1-\bar{y})<0$ and because of $y_1<\bar{y}$ this implies $\lambda_0>0$ and therefore we always may assume $\lambda_0=1$.

Hence

$$\max_{M_k}(y(\mu)+\lambda_1x_1(\mu)+\cdots+\lambda_nx_n(\mu))=\bar{y}$$

which proves the Lemma.

The implications of Lemmas 1 and 2 for constrained extremal problems in V_k are given in the following

THEOREM 1. *Let the functions* $X_j(\theta)$, $j = 1, \cdots, n$, *be real-analytic on* $[0, 2\pi]$ *and let them together with the constant* 1 *be linearly independent. Let the functional* Φ *on* V_k *admit Gâteaux derivatives* $L = L_f$ *for each* f *of* V_k *such that the corresponding indicators* J *are never a linear combination of the* X_j *and the constant* 1. *If* f_0 *maximizes* $\operatorname{Re}\Phi$ *over* V_k *under the side conditions*

$$\int X_j(\theta)d\mu = \xi_j, \qquad j = 1, \cdots, n, \tag{14}$$

where $(\xi_1, \cdots, \xi_n)$ *is an inner point of* A (cf. (13)), *then its representing measure* μ_0 *has finite support, its total variation* $\|\mu_0\|$ *is* k (*i.e.* f_0 *has boundary rotation* k), *and there are multipliers* $\lambda_1, \cdots, \lambda_n$ *such that on the disjoint supports of the upper and of the lower variation of* μ_0 *the function*

$$Y = J + \lambda_1 X_1 + \cdots + \lambda_n X_n$$

reaches its maximum and its minimum respectively, i.e. we have the relations (12) *with* J *replaced by* Y.

PROOF. From the section preceding Lemma 1 it follows that $\int J(\theta)d(\mu - \mu_0) \leqq 0$ for each measure μ of M_k satisfying the constraints (14), because $\mu_\varepsilon = \mu_0 + \varepsilon(\mu - \mu_0)$, $0 \leqq \varepsilon \leqq 1$, satisfies (14) if μ_0 and μ do so. This implies that μ_0 maximizes the functional $\int J(\theta)d\mu$ over M_k under the conditions (14). Being this way within the frame of Lemma 2 the conclusion of Theorem 1 follows at once, since Y is real-analytic and non-constant.

3. The range of $\log f'(a)$

To each f of V_k corresponds the function $\log f'(z)$, which is holomorphic in the unit disc **D** and vanishes at the origin. For a fixed $a \in \mathbf{D}$ we consider the range

$$R_k(a) = \{\log f'(a) : f \in V_k\}.$$

Without loss of generality we may suppose a to be positive. Since M_k is convex and

$$\log f'(a) = \int -\log(1 - ae^{i\theta})d\mu, \qquad \mu \in M_k,$$

the range $R_k(a)$ is convex as well, and it contains the origin since the identity mapping belongs to V_k. To determine the boundary of $R_k(a)$ it is therefore sufficient to maximize

$$\mathrm{Re}\{e^{-i\phi}\log f'(a)\} \tag{15}$$

within V_k for each $\phi \in \mathbf{R}$.

The corresponding indicator J is given by

$$J(\theta)=\mathrm{Re}\{-e^{-i\phi}\log(1-ae^{i\theta})\}.$$

The curve

$$\theta \to \log(1-ae^{i\theta}), \qquad 0 \leqq \theta \leqq 2\pi,$$

is convex and so the support of the extremal measure μ_0 consists of exactly two points: μ_0^+ and μ_0 are concentrated at the points where J reaches its unique maximum and minimum, respectively. At these points we have

$$\frac{dJ}{d\theta}=\mathrm{Re}\left\{e^{-i\phi}\frac{iae^{i\theta}}{1-ae^{i\theta}}\right\}=\frac{a}{|1-ae^{i\theta}|^2}\mathrm{Im}\{e^{-i\phi}(e^{i\theta}-a)\}$$

and they are obtained by intersecting the unit circle with the straight line $\{a+\lambda e^{i\phi}:\lambda \in \mathbf{R}\}$.

THEOREM 2. *For each a of* $(0,1)$ *the range*

$$R_k(a)=\{\log f'(a): f \in V_k\}$$

is convex and its boundary is given by the curve

$$\theta \to -(k/2+1)\log(1-ae^{i\theta})+(k/2-1)\log(1-ae^{i\theta'}), \quad \theta \in |0,2\pi|$$

where $e^{i\theta}$ and $e^{i\theta'}$ lie on a straight line through a.

For $\phi=0$ in (15) the above theorem is due to Loewner [6] and for $\phi=\pi/2$ to Paatero [8]. For general ϕ see [10] and [11] as well.

Now we consider the problem of maximizing $\mathrm{Re}\{\log f'(a)\}$ under the side condition $f''(0)=0$, i.e. to determine a function f_0 in V_k such that $f_0''(0)=0$ and

$$\begin{aligned}\mathrm{Re}\{\log f_0'(a)\} &= \max \mathrm{Re}\{\log f'(a): f \in V_k, f''(0)=0\}\\ &= \max\left\{-\frac{1}{2}\int \log(1+a^2-2a\cos\theta)d\mu : \mu \in M_k, \int e^{i\theta}d\mu=0\right\}.\end{aligned} \tag{16}$$

Note that $f''(0)=0$ for all odd functions of V_k.

The indicator corresponding to $L=\mathrm{Re}\{\log f'(a)\}$ is given by

$$J(\theta)=-\tfrac{1}{2}\log(1+a^2-2a\cos\theta\}.$$

According to Lemma 2 there are multipliers λ_j such that with

$$Y(\theta) = J(\theta) + \lambda_1 \cos\theta + \lambda_2 \sin\theta$$

we have

$$\max_{M_k} \int Y(\theta) d\mu = \int J(\theta) d\mu_0, \tag{17}$$

where μ_0 is the representing measure of f_0. Referring to the proof of Lemma 2, with $\cos\theta = X_1(\theta)$ and $\sin\theta = X_2(\theta)$, the functionals $y(\mu)$ and $x_1(\mu)$ remain unchanged while $x_2(\mu)$ changes sign if $\mu(\theta)$ is replaced by $\bar{\mu}(\theta) = \mu(2\pi - \theta)$. Therefore the body B is symmetric to the plane $x_2 = 0$ and consequently we may choose $\lambda_2 = 0$. Hence $\operatorname{supp} \mu_0^+$ and $\operatorname{supp} \mu_0$ are subsets of $[0, 2\pi]$, where

$$Y(\theta) = J(\theta) + \lambda_1 \cos\theta$$

attains its maximum and minimum respectively. The derivative

$$\frac{dY}{d\theta} = -\sin\theta \cdot \left(\frac{a}{1 + a^2 - 2a\cos\theta} + \lambda_1 \right)$$

vanishes for $\theta = 0$ and $\theta = \pi$ and possibly for two other points α and $-\alpha$, where $0 < \alpha < \pi$ and $a + \lambda_1(1 + a^2 - 2a\cos\alpha) = 0$. If 0 and π were the only zeros, $Y(\theta)$ would have the only maximum at $\theta = 0$ and the only minimum at $\theta = \pi$, or vice-versa, and this would violate the side condition $\int \cos\theta d\mu_0 = 0$. Hence there are two further zeros α and $-\alpha$. Since the second derivative $d^2Y/d\theta^2$ is positive for $\theta = \pm\alpha$, these are the points where Y attains its minimum. Local maxima and minima being alternating it follows that $\theta = 0$ and $\theta = \pi$ are maxima. But μ_0^+ cannot be carried by $\theta = 0$ or $\theta = \pi$ alone, because of the restriction $\int \cos\theta d\mu_0 = 0$. Both points have to be absolute maxima and the equations $Y(0) = Y(\pi)$ and $dY/d\theta\,(\alpha) = 0$ determine λ_1 and α:

$$\cos\alpha = \frac{1 + a^2}{2a} - \frac{1}{\log\dfrac{1+a}{1-a}}, \qquad \frac{a}{2} < \cos\alpha < a. \tag{18}$$

The side conditions $\int \cos\theta d\mu_0 = \int \sin\theta d\mu_0 = 0$ imply that μ_0^- has equal masses $\frac{1}{2}(k/2 - 1)$ at $\theta = \pm\alpha$ and that μ_0^+ has masses $(k/2 + 1)\cdot m$ and $(k/2+1)\cdot(1-m)$ at the points $\theta = 0$ and $\theta = \pi$ respectively, where

$$m = \frac{1}{2}\left(1 + \frac{k-2}{k+2}\cos\alpha\right), \qquad m \in \left(\frac{1}{2}, \frac{1+a}{2}\right). \tag{19}$$

It is interesting to observe that α depends on a but not on k.

THEOREM 3. *Let the function f_0 maximize* $\operatorname{Re}\{\log f'(a)\}$ *among those functions f of V_k that satisfy the side condition $f''(0)=0$. Then f_0 is unique and*

$$f_0'(z)=\frac{[(1-e^{i\alpha}z)(1-e^{-i\alpha}z)]^{k/4-1/2}}{[(1-z)^m(1+z)^{1-m}]^{k/2+1}},$$

where α and m are defined by (18) *and* (19).

REMARKS. (1) The function f_0 of this Theorem is odd if and only if $k=2$, i.e. in this case the extremality of $|f'(a)|$ under the condition $f''(0)=0$ forces f_0 to be odd, but no more if $k>2$.

(2) In [9] J. A. Pfaltzgraff and B. Pinchuk introduced the class Λ_k of meromorphic functions of boundary rotation $\leqq k$. These are functions represented in **D** by $f(z)=1/z+b_0+b_1z+\cdots$ such that

$$f'(z)\neq 0 \quad\text{and}\quad \lim_{r\to 1}\int\left|d_\theta \arg\frac{d}{d\theta}f(re^{i\theta})\right|\leqq k.$$

For a fixed a, $0<a<1$, these authors considered the problem of maximizing and minimizing $|f'(a)|$ over Λ_k. They have shown that $f\in\Lambda_k$ if and only if

$$f'(z)=-\frac{1}{z^2}\exp\int\log(1-ze^{i\theta})d\mu,$$

where $\mu\in M_k$ and $\int e^{i\theta}d\mu=0$, and, using a Golusin type variation, have shown that the extremal measures have to be located on at most 4 points of $[0,2\pi]$. Using this fact J. Noonan [7] has completely determined these measures with a method working in this specific situation.

The problem actually reduces to maximizing and minimizing $\int\log|1-ae^{i\theta}|d\mu$ over M_k under the side condition $\int e^{i\theta}d\mu=0$. The minimum part of it has been also solved above and the maximum can be handled in a similar, although slightly more complicated way. Along the same line one can solve the same problem under a side condition $\int e^{i\theta}d\mu=r$, where r is a given number of the interval $(-k,k)$.

4. Vanishing first coefficients

Let us recall that $f\in V_k$ if and only if

$$1+\frac{z\cdot f''(z)}{f'(z)}=\frac{1}{2}\int\frac{1+e^{i\theta}z}{1-e^{i\theta}z}d\mu$$

$$=1+c_1z+c_2z^2+\cdots,$$

where

$$c_n = \int e^{in\theta} d\mu \quad \text{and} \quad \mu \in M_k.$$

Comparing coefficients we get

$$\begin{aligned} &2 \cdot 1 \cdot a_2 = c_1, \\ (20) \qquad &3 \cdot 2 \cdot a_3 = c_2 + 2a_2 \cdot c_1, \\ &n \cdot (n-1) a_n = c_{n-1} + 2a_2 c_{n-2} + \cdots + (n-1) a_{n-1} c_1. \end{aligned}$$

Thus

$$a_2 = \cdots = a_m = 0 \tag{21}$$

if and only if

$$c_1 = \cdots = c_{m-1} = 0,$$

and in this case the recursion formula (20) simplifies to

$$\begin{aligned} n(n-1) \cdot a_n &= c_{n-1}, \qquad m < n \leqq 2m \\ &= c_{2m} + \frac{1}{m} \cdot c_m^2, \qquad n = 2m+1. \end{aligned}$$

Schiffer and Tammi [12] have shown by the help of their variational method, that for $n = m+1$ the maximum of $\operatorname{Re} a_n$, under the restrictions (21) is attained by the symmetric function

$$f'(z) = \left[\frac{(1+z^{n-1})^{k/2-1}}{(1-z^{n-1})^{k/2+1}} \right]^{1/n-1} \tag{22}$$

Hence

$$f(z) = z + \frac{k}{n(n-1)} z^n + \cdots.$$

The next theorem shows that this remains true for all n with $m < n \leqq 2m$ and that the situation changes completely if $n = 2m+1$.

THEOREM 4. *Let $f \in V_k$ satisfy the conditions* (21):

$$f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots, \qquad m \geqq 2.$$

Then

$$\operatorname{Re} a_n \leqq \frac{k}{n(n-1)}, \qquad \text{if } m < n \leqq 2m,$$

and equality occurs for the $(n-1)$-fold symmetric function f given by (22).

If $n=2m+1$ then

$$2m(2m+1)\operatorname{Re} a_n \leqq 2+k^2/m \qquad \textit{for } m \leqq k/2 \qquad \textit{and}$$

$$2m(2m+1)\operatorname{Re} a_n \leqq k+\frac{(k+2)^2}{4m-(k-2)} \qquad \textit{for } m \geqq k/2.$$

In the first case equality occurs for the function f_1 with derivative

$$f_1'(z)=\left[\frac{(1+z^m)^{k/2-1}}{(1-z^m)^{k/2+1}}\right]^{1/m}$$

$$=1+\frac{k}{m}z^m+\frac{1}{2m}\left(2+\frac{k^2}{m}\right)z^{2m}+\cdots$$

and for the function $-f_1(-z)$. In the second case we have equality for the functions $f_2(z)$ and $-f_2(-z)$, where

$$f_2'(z)=\left[\frac{(1-e^{i\alpha}z^m)(1-e^{-i\alpha}z^m)^{k/4-1/2}}{(1-z^m)^{k/2+1}}\right]^{1/m}$$

$$=1+\frac{1}{m}\left[\left(\frac{k}{2}+1\right)-\left(\frac{k}{2}-1\right)\cos\alpha\right]z^m$$

$$+\frac{1}{2m}\left[k+\frac{(k+2)^2}{4m-(k-2)}\right]z^{2m}+\cdots$$

and α is given by

$$\cos\alpha=\frac{k+2}{k-2-4m}, \qquad \alpha\in(\pi/2,\pi).$$

REMARK. Let $f(z)=1/z+a_0+a_1z+\cdots+a_nz^n+\cdots$ be a function of the class Λ_k as defined in Remark (2) following Theorem 3. Noonan [7] has shown that $|a_1|\leqq k/2$ and $|a_2|\leqq k/6$, and that these bounds are sharp. To determine the maximum of $|a_3|$ over Λ_k it is sufficient to find min Re a_3 within this class. From $z^2\cdot f'(z)=-\exp(-g(z))$, with

$$g(z)=\tfrac{1}{2}c_2z^2+\tfrac{1}{3}c_3z^3+\tfrac{1}{4}c_4z^4+\cdots \qquad \text{and}$$

$$c_j=\int e^{ij\theta}d\mu, \qquad \mu\in M_k, \quad j=1,2,\cdots,$$

it follows

$$3a_3=\tfrac{1}{4}(c_4-\tfrac{1}{2}c_2^2).$$

It thus remains to maximize $\operatorname{Re}\{\int e^{4i\theta} - \frac{1}{2}(\int e^{2i\theta} d\mu)^2\}$ over M_k under the side condition $\int e^{i\theta} d\mu = 0$. As it turns out in the following proof, this is just the problem of the above theorem for $m = 2$ and $n = 5$.

PROOF. (1) Case: $m < n \leqq 2m$

This case can be settled without any variational method: with $\mu = (k/2+1)\nu - (k/2-1)\nu'$ (cf. (11)) we have

$$\begin{aligned} n(n-1)\operatorname{Re} a_n &= \int \cos(n-1)\theta d\mu \\ &= (k/2+1)\int \cos(n-1)\theta d\nu - (k/2-1)\int \cos(n-1)\theta d\nu' \\ &\leqq k/2+1+(k/2-1) = k. \end{aligned}$$

Equality occurs if and only if ν and ν' are concentrated on the sets

$$\left\{\theta_{2j} = \frac{2j}{n-1}\pi\right\} \quad \text{and} \quad \left\{\theta_{2j+1} = \frac{2j+1}{n-1}\pi\right\}, \qquad j = 0, \cdots, n-2,$$

respectively. For equally distributed measures we get (22), and the first part is proved.

For any probability measures $\nu = \{(\theta_{2j}, \nu_j)\}$ and $\nu' = \{(\theta_{2j+1}, \nu'_j)\}$, $j = 0, \cdots, n-2$, on the above sets, the conditions $c_1 = \cdots = c_m = 0$ together with $\Sigma \nu_j = 1$ and $\Sigma \nu'_j = 1$ represent a linear system which admits always the solution $\nu_j = \nu'_j = 1/(n-1)$, but there is only this one if $n = m+1$.

(2) Case: $n = 2m+1$

Here we have to maximize

$$\begin{aligned} (2m+1)2m \operatorname{Re} a_{2m+1} &= \operatorname{Re}\left\{\int e^{i2m\theta} d\mu + \frac{1}{m}\left(\int e^{im\theta} d\mu\right)^2\right\} \\ &= \int \cos 2m\theta \, d\mu + \frac{1}{m}\left(\int \cos m\theta \, d\mu\right)^2 \\ &\quad - \frac{1}{m}\left(\int \sin m\theta \, d\mu\right)^2, \end{aligned}$$

under the restrictions

$$c_j = \int e^{ij\theta} d\mu = 0, \qquad j = 1, \cdots, m-1. \tag{23}$$

The integral $\int \sin m\theta d\mu$ has to vanish in the extremal case, because otherwise the measure $\mu_0 = \frac{1}{2}(\mu_0(\theta) + \mu_0(2\pi - \theta))$ would yield a bigger value. Thus it remains to maximize

$$R(\mu) = \int \cos 2m\theta \, d\mu + \frac{1}{m}\left[\int \cos m\theta \, d\mu\right]^2$$

under the restrictions (23). The indicator J corresponding to the functional $R(\mu)$ is given by

$$J(\theta) = \cos 2m\theta + c \cdot \cos m\theta, \qquad \text{where } c = \frac{2}{m}\int \cos m\theta d\mu_0.$$

Using complex notation Lemma 2 and Theorem 1 imply that there are complex multipliers $\lambda_1, \cdots, \lambda_{m-1}$ such that

$$\max_{M_k}[y(\mu) + \mathrm{Re}\{\lambda_1 c_1(\mu) + \cdots + \lambda_{m-1}c_{m-1}(\mu)\}] = \bar{y},$$

where $y(\mu) = \int J(\theta)d\mu$ and

$$\bar{y} = \max\{y(\mu) \mid \mu \in M_k, c_j(\mu) = 0, j = 1, \cdots, m-1\}.$$

Let $\varepsilon = e^{2\pi i/m}$ and let μ_k be obtained from a measure μ by setting $\mu_k(\theta) = \mu(\theta + 2\pi k/m)$, $k = 1, 2, \cdots, m-1$. Since J remains unchanged and the $c_j(\mu)$ get multiplied by ε^{jk} when μ is replaced by μ_k, it follows that all the points $(y, c_1\varepsilon^k, \cdots, c_{m-1}\varepsilon^{k(m-1)})$, $k = 0, 1, \cdots, m-1$, belong to the body B considered in the proof of Lemma 2, if one of them does so. Adding the inequalities

$$y + \mathrm{Re}\left\{\sum_{j=1}^{m-1} \lambda_j c_j \varepsilon^{kj}\right\} \leqq \bar{y}, \qquad k = 0, 1, \cdots, m-1,$$

implies $y(\mu) \leqq \bar{y}$ for $\mu \in M_k$ and because of $\max_{M_k} y(\mu) \geqq \bar{y}$ we conclude that $\max_{M_k} y(\mu) = \bar{y}$, i.e. the extremal measures of our problem are among those which maximize $R(\mu)$ over M_k. Thus we first may handle the problem as if there were no constraints and then single out those extremal measures which satisfy the side conditions (23).

The maximum of $R(\mu)$ over M_k was determined by R. J. Leach (cf. [4]). Using a stepfunction method he showed that

$$R(\mu) \leqq 2 + \frac{k^2}{m} \qquad \text{for } m \leqq k/2 \qquad \text{and}$$

$$R(\mu) \leqq k + \frac{(k+2)^2}{4m - (k-2)} \quad \text{for } m \geqq k/2,$$

and that the functions f_1 and f_2 in Theorem 4 are extremal respectively.

5. In order to get all extremal functions, we consider the general problem of maximizing the functional

$$R(\mu)=\mathrm{Re}\left\{\int e^{i2m\theta}d\mu+\lambda\cdot\left(\int e^{im\theta}d\mu\right)^2\right\}$$

$$=\int \cos 2m\theta d\mu+\lambda\left(\int \cos m\theta d\mu\right)^2-\lambda\left(\int \sin m\theta d\mu\right)^2$$

over M_k without any constraints, where λ is a real number and m a positive integer.

For $m=1$ this problem has been solved by Kirwan and Schober [3] as well as by Lehto and Tammi (cf. [5] on page 79) with different methods.

The case $\lambda=0$ is trivial. We have $R(\mu)\leqq k$ and equality occurs if and only if μ^+ and μ^- (the positive and negative variations of μ) are carried by the sets

(24) $$0,\frac{\pi}{m},\cdots,\frac{m-1}{m}\pi \quad\text{and}\quad \frac{\pi}{2m},\frac{3\pi}{2m},\cdots,\frac{2m-1}{m}\pi$$

respectively. The distribution of μ^+ and μ^- within these sets is irrelevant.

If $\lambda>0$, we get

(25) $$\int \sin m\theta d\mu_0=0$$

as a first necessary condition for an extremal measure μ_0, and it remains to maximize

$$R_0(\mu)=\int \cos 2m\theta\, d\mu+\lambda\left(\int \cos m\theta\, d\mu\right)^2$$

over M_k. Lemma 1 provides a second necessary condition, namely

(26) $$\int J(\theta)d(\mu-\mu_0)\leqq 0 \qquad \text{for all } \mu \text{ of } M_k,$$

where

(27) $$J(\theta)=\cos 2m\theta+c\cdot\cos m\theta \quad\text{and}\quad c=2\lambda\cdot\int \cos m\theta\, d\mu_0.$$

According to this lemma, μ_0 has to be concentrated on the zeros of

$$dJ/d\theta=-m\cdot\sin m\theta\cdot(4\cos m\theta+c).$$

It is sufficient to consider the case $c\geqq 0$, because the equation $\mu_0'(\theta)=$

$\mu_0(\theta + \pi/m)$ establishes a one-to-one relation between the measures μ_0 corresponding to some c and those which correspond to $c' = -c$.

Obviously there are the two sets

$$\left\{\frac{\pi}{m}\cdot 2j\right\}_{j=0,\cdots,m-1} \quad \text{and} \quad \left\{\frac{\pi}{m}(2j+1)\right\}_{j=0,\cdots,m-1} \tag{28}$$

of zeros, and if $|c| \geqq 4$, these are the only ones. J reaches its maximum on the first set and its minimum on the second one. Hence μ_0^+ and μ_0^- have to be spread over these two sets respectively in a completely arbitrary way, and we get the first bound

$$R(\mu_0) = 2 + \lambda \cdot k^2 \quad \text{and} \quad c = 2\lambda k.$$

Since $c \geqq 4$, this extremal measure can occur only when $\lambda k \geqq 2$.

If $0 \leqq c < 4$, there is a unique α in $(0, \pi)$ satisfying $4\cos\alpha + c = 0$ and we have the further zeros

$$\pm\frac{\alpha}{m}, \frac{2\pi \pm \alpha}{m}, \ldots, \frac{2\pi\cdot(m-1)\pm\alpha}{m}. \tag{29}$$

If $c = 0$, then $\alpha = \pi/2$ and J reaches its maximum and its minimum on the first and the second set of (24) respectively, i.e. we are in the case of $\lambda = 0$. Hence, since $\lambda > 0$ now, we have $c > 0$. In this case J attains its maximum on the first set of (28) and its minimum on the set (29). Lemma 1 shows that μ_0^+ and μ_0^- have to be spread over these two sets respectively. From (27) and $4\cos\alpha + c = 0$ we get the equalities

$$\begin{gathered} c = 2\lambda\left(\frac{k}{2}+1-\left(\frac{k}{2}-1\right)\cos\alpha\right), \qquad \cos\alpha = \frac{\lambda(k+2)}{\lambda(k-2)-4} \qquad \text{and} \\ R_0(\mu_0) = (k/2+1)-(k/2-1)\cos 2\alpha + \lambda(k/2+1-(k/2-1)\cos\alpha)^2 \end{gathered} \tag{30}$$

for each of these distributions. If μ_0^- is spread over (29) in such a way that the two subsets corresponding to $+\alpha$ and to $-\alpha$ carry the same weight $\frac{1}{2}(k/2-1)$, condition (25) is satisfied as well and μ_0 maximizes $R(\mu)$ for the chosen c, $0 < c < 4$. Indeed, the second equation of (30) shows that these extremal measures only occur when $0 < \lambda k < 2$.

Thus we determined the set of extremal measures for $\lambda > 0$.

By setting $m\theta = \pi/2 + m\phi$ in case of $\lambda < 0$ we reduce the problem to maximize

$$R_0(\mu) = -\int \cos 2m\phi \, d\mu + |\lambda| \left(\int \cos m\phi \, d\mu\right)^2$$

over M_k. Here the indicator is $J(\phi) = -\cos 2m\phi + c \cos m\phi$ where $c = 2|\lambda| \int \cos m\phi d\mu_0$ and μ_0 is an extremal measure. This leads to a discussion very similar to the one before.

In going back to the situation of $n = 2m + 1$ in Theorem 4 we choose for μ_0^+ and μ_0^- equally distributed measures over the first set of (28) and over (29) respectively. The measure $\mu_0 = \mu_0^+ - \mu_0^-$ then will be extremal for $R(\mu)$ and the corresponding functions $f_1(z)$ (for $2m \leqq k$) and $f_2(z)$ (for $2m \geqq k$) will be extremal for $\operatorname{Re} a_n$. They are m-fold symmetric and together with the functions $-f_1(-z)$ and $-f_2(-z)$ they are the only m-fold symmetric extremals for $\operatorname{Re} a_n$. But there are many more extremal measures satisfying the constraints (23), if $m \geqq 3$.

References

1. R. Boutellier, *Einige Extremalprobleme sternförmiger Funktionen*, Dissertation Nr. 6322 der ETH, 1979.

2. W. E. Kirwan and G. Schober, *On extreme points and support points for some families of univalent functions*, Duke Math. J. **42** (1975), 285–296.

3. W. E. Kirwan and G. Schober, *Inverse coefficients for functions of bounded boundary rotation*, J. Analyse Math. **36** (1979), 167–178.

4. R. J. Leach, *Coefficients of symmetric functions of bounded boundary rotation*, Canad. J. Math. **26** (1974), 1351–1355.

5. O. Lehto, *Univalent Functions, Schwarzian Derivatives and Quasiconformal Mappings*, Contributions to Analysis, Monograph No. 27 de l'Enseignement mathématique, Genève, 1979, pp. 73–83.

6. K. Loewner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I*, Math. Ann. **89** (1923), 103–121.

7. J. Noonan, *Meromorphic functions of bounded boundary rotation*, Michigan J. Math. **18** (1971), 343–353.

8. V. Paatero, *Ueber die konforme Abbildung von Gebieten, deren Ränder von beschränkter Drehung sind*, Ann. Acad. Sci. Fenn. **A33** (1931).

9. J. A. Pfaltzgraff and B. Pinchuk, *A variational method for classes of meromorphic functions*, J. Analyse Math. **24** (1971), 101–150.

10. A. Pfluger, *Functions of bounded boundary rotation and convexity*, J. Analyse Math. **30** (1976), 437–451.

11. B. Pinchuk, *A variational method for functions of bounded boundary rotation*, Trans. Amer. Math. Soc. **138** (1969), 107–113.

12. M. Schiffer and O. Tammi, *A method of variations for the functions with bounded boundary rotation*, J. Analyse Math. **17** (1966), 109–144.

13. G. Schober, *Univalent Functions — Selected Topics*, Lecture Notes in Math., Vol. 478, Springer Verlag, 1975.

Imperial College
London, England

Eidg. Techn. Hochschule
CH-8092 Zurich, Switzerland